[1]

DIMER SPACES AND GLIDING SYSTEMS

VLADIMIR TURAEV

Abstract. Dimer coverings (or perfect matchings) of a finite graph are classical objects of graph theory appearing in the study of exactly solvable models of statistical mechanics. We introduce more general dimer labelings which form a topological space called the dimer space of the graph. This space turns out to be a cubed complex whose vertices are the dimer coverings. We show that the dimer space is nonpositively curved in the sense of Gromov, so that its universal covering is a CAT(0)-space. We study the fundamental group of the dimer space and, in particular, obtain a presentation of this group by generators and relations. We discuss connections with right-angled Artin groups and braid groups of graphs. Our approach uses so-called gliding systems in groups designed to produce nonpositively curved cubed complexes.

1. Introduction

A dimer covering, or a perfect matching, of a graph is a set of edges such that each vertex is incident to exactly one of them. Dimer coverings have been extensively studied since 1960's in connection with exactly solvable models of statistical mechanics and, more recently, in connection with path algebras, see [Bo], [Ke] and references therein. In the present paper we develop a geometric approach to dimer coverings. Specifically, we introduce and study a dimer space and a dimer group of a finite graph.

Consider a finite graph Γ (without loops but possibly with multiple edges). A *dimer labeling* of Γ is a labeling of the edges of Γ by non-negative real numbers such that for every vertex of Γ, the labels of the adjacent edges sum up to give 1 and only one or two of these labels may be non-zero. The set $L = L(\Gamma)$ of dimer labelings of Γ is a closed subset of the cube formed by all labelings of edges by numbers in $[0, 1]$. We endow L with the induced topology. We will show that L is a CW-complex, possibly disconnected, and all connected components of L are aspherical, i.e., have trivial higher homotopy groups.

The characteristic function of a dimer covering of Γ, carrying the edges of the covering to 1 and all other edges to 0, is a dimer labeling. The points of L represented in this way by dimer coverings lie in a single component $L_0 = L_0(\Gamma)$ of L called the *dimer space*. The fundamental group of L_0 is the *dimer group* of Γ. We show that the dimer group is torsion-free, residually nilpotent, residually finite, biorderable, biautomatic, has solvable word and conjugacy problems, satisfies the Tits alternative, embeds in $SL_n(\mathbb{Z})$ for some n, and embeds in a finitely generated right-handed Artin group. The fundamental groups of other components of L are isomorphic to the dimer groups of certain subgraphs of Γ and share the properties listed above.

[1]AMS Subject Classification: 57M15, 05C10, 20F36, 20F67, 82B20

The reader familiar with geometric group theory will immediately recognize the properties of groups arising in the study of Cartan-Alexandrov-Toponogov (0)-spaces in the sense of Gromov, briefly called CAT(0)-spaces. Such a space has a geodesic metric in which all geodesic triangles are at least as thin as triangles in $\mathbb{R}^2$ with the same sides. The theory of CAT(0)-spaces will be our main tool. We show that L_0 has a natural structure of a cubed complex whose vertices are the dimer coverings of Γ. This cubed complex satisfies Gromov's link condition, and therefore L_0 is nonpositively curved. The universal covering of L_0 is a CAT(0)-space. Similar results hold for other components of L.

The cubed structure in L_0 arises from the following observation. Consider an embedded circle s in Γ of even length (i.e., formed by an even number of edges). Suppose that every second edge of s belongs to a dimer covering A of Γ. Removing these edges from A and adding instead all other edges of s we obtain a new dimer covering sA. We say that sA is obtained from A by *gliding* along s. Deforming the labeling determined by A into the labeling determined by sA we obtain a path (a 1-cube) in L_0. More generally, we can start with a family of $k \geq 1$ disjoint embedded circles in Γ of even length meeting A along every second edge. Gliding A along (some of) these circles, we obtain 2^k dimer coverings of Γ that serve as the vertices of a k-dimensional cube in L_0. Deforming the labelings associated with these vertices we obtain all points of the cube. Similar ideas were introduced in [STCR] in the study of domino tilings of planar regions.

Besides the properties of the dimer group mentioned above, we obtain a presentation of this group by generators and relations. Denote the set of dimer coverings of Γ by $\mathcal{D}$. For $A \in \mathcal{D}$ and a vertex v of Γ, denote the only edge of A incident to v by A_v. We say that a triple of dimer coverings $A, B, C \in \mathcal{D}$ is *flat* if for any vertex v of Γ, at least two of the edges A_v, B_v, C_v coincide. For each $A_0 \in \mathcal{D}$, the dimer group $\pi_1(L_0, A_0)$ is generated by the symbols $\{y_{A,B}\}_{A,B\in\mathcal{D}}$ numerated by ordered pairs of dimer coverings. The defining relations are as follows: $y_{A,C} = y_{A,B}\, y_{B,C}$ for any flat triple $A, B, C \in \mathcal{D}$ and $y_{A_0,A} = 1$ for all $A \in \mathcal{D}$. We obtain a similar presentation for the fundamental groupoid of the pair $(L_0, \mathcal{D})$.

Among other results of the paper note a connection of the dimer groups to the braid groups of graphs and a generalization of the dimer groups to hypergraphs.

The study of dimer coverings suggests an axiomatic framework of gliding systems. A *gliding system* in a group G consists of certain elements of G called *glides* and a binary relation on the set of glides called *independence* satisfying a few axioms. Given a gliding system in G and a set $\mathcal{E} \subset G$, we construct a cubed complex $X_\mathcal{E}$ called the *glide complex*. The fundamental groups of the components of $X_\mathcal{E}$ are the *glide groups*. We formulate conditions ensuring that $X_\mathcal{E}$ is nonpositively curved. One can view gliding systems as devices producing nonpositively curved complexes and interesting groups. The dimer space of a graph is an instance of the glide complex where G is the group of $\mathbb{Z}/2\mathbb{Z}$-valued function on the set of edges, the glides are the characteristic functions of the sets of edges forming embedded circles of even length, the independence mirrors the disjointness of embedded circles, and $\mathcal{E}$ consists of the characteristic functions of dimer coverings.

The paper is organized as follows. Sections 2–7 are concerned with glides. We define gliding systems (Section 2), construct the glide complexes (Section 3), study natural maps between the glide groups (Section 4), embed the glide groups into right-angled Artin groups (Section 5), produce presentations of the glide groups by

generators and relations (Section 6), and study a class of set-like gliding systems (Section 7). Next, we introduce and study dimer complexes (Section 8) and dimer groups (Section 9). In Section 10 we discuss connections with braid groups. In Section 11 we extend our definitions and results to hypergraphs.

The author would like to thank M. Ciucu for several stimulating discussions. This work was partially supported by the NSF grant DMS-1202335.

2. Glides

2.1. **Gliding systems.** By a *gliding system* in a group G we mean a pair of sets $(\mathcal{G} \subset G \setminus \{1\}, \mathcal{I} \subset \mathcal{G} \times \mathcal{G})$ satisfying the following conditions:

(1) if $s \in \mathcal{G}$, then $s^{-1} \in \mathcal{G}$;

(2) if $(s,t) \in \mathcal{I}$ with $s,t \in \mathcal{G}$, then $st = ts$ and $(s^{-1}, t) \in \mathcal{I}$, $(t,s) \in \mathcal{I}$;

(3) $(s,s) \notin \mathcal{I}$ for all $s \in \mathcal{G}$.

The elements of $\mathcal{G}$ are called *glides*. The inverse of a glide is a glide while the unit $1 \in G$ is never a glide. We say that two glides s,t are *independent* if $(s,t) \in \mathcal{I}$. By (2), the independence is a property of non-ordered pairs of glides preserved under inversion of one or both glides. A glide is never independent from itself or its inverse. Also, independent glides commute.

For $s \in \mathcal{G}$ and $A \in G$, we say that $sA \in G$ is obtained from A by *(left) gliding along* s. One can similarly consider right glidings but we do not need them. Clearly, the gliding of sA along s^{-1} yields $s^{-1}sA = A$.

A *set of independent glides* is a set $S \subset \mathcal{G}$ such that $(s,t) \in \mathcal{I}$ for any distinct $s,t \in S$. Since independent glides commute, a finite set of independent glides S determines an element $[S] = \prod_{s \in S} s$ of G. By definition, $[\emptyset] = 1$.

2.2. **Examples.** 1. For any group G, the following pair is a gliding system: $\mathcal{G} = G \setminus \{1\}$ and $\mathcal{I}$ is the set of all pairs $(s,t) \in \mathcal{G} \times \mathcal{G}$ such that $s \neq t^{\pm 1}$ and $st = ts$.

2. The pair $(\mathcal{G} = G \setminus \{1\}, \mathcal{I} = \emptyset)$ is a gliding system in a group G.

3. Let G be a free abelian group with free commuting generators $\{g_i\}_i$. Then

$$\mathcal{G} = \{g_i^{\pm 1}\}_i, \quad \mathcal{I} = \{(g_i^{\varepsilon}, g_j^{\mu}) \,|\, \varepsilon = \pm 1, \mu = \pm 1, i \neq j\}$$

is a gliding system in G.

4. A generalization of the previous example is provided by the theory of right-angled Artin groups (see [Ch] for an exposition). A right-angled Artin group is a group allowing a presentation by generators and relations in which all relators are commutators of the generators. Any graph Γ with the set of vertices V determines a right-angled Artin group $G = G(\Gamma)$ with generators $\{g_s\}_{s \in V}$ and relations $g_s g_t g_s^{-1} g_t^{-1} = 1$ which hold whenever $s,t \in V$ are connected by an edge in Γ (we write then $s \leftrightarrow t$). Abelianizing G we obtain that $g_s \neq g_t^{\pm 1}$ for $s \neq t$. The pair

$$\mathcal{G} = \{g_s^{\pm 1}\}_{s \in V}, \quad \mathcal{I} = \{(g_s^{\varepsilon}, g_t^{\mu}) \,|\, \varepsilon = \pm 1, \mu = \pm 1, s,t \in V, s \neq t, s \leftrightarrow t\} \tag{2.2.1}$$

is a gliding system in G.

5. Let E be a set and $G = 2^E$ be the power set of E consisting of all subsets of E. We define multiplication in G by $AB = (A \cup B) \setminus (A \cap B)$ for $A, B \subset E$. This turns G into an abelian group with unit $1 = \emptyset$, the *power group of* E. Clearly, $A^{-1} = A$ for all $A \in G$. Pick any set $\mathcal{G} \subset G \setminus \{1\}$ and declare elements of $\mathcal{G}$ independent when they are disjoint as subsets of E. This gives a gliding system in G.

6. Let E be a set, H be a multiplicative group, and $G = H^E$ be the group of all maps $E \to H$ with pointwise multiplication. The *support* of a map $f : E \to H$

is the set $f^{-1}(H \setminus \{1\}) \subset E$. Pick a set $\mathcal{G} \subset G \setminus \{1\}$ invariant under inversion and declare elements of $\mathcal{G}$ independent if their supports are disjoint. This gives a gliding system in G. When H is a cyclic group of order 2, we recover Example 5 via the isomorphism $H^E \cong 2^E$ carrying a map $E \to H$ to its support.

3. Glide complexes and glide groups

We discuss cubed complexes associated with gliding systems. We begin with generalities on cubed complexes referring for details to [BH], Chapters I.7 and II.5.

3.1. **Cubed complexes.** Set $I = [0, 1]$. A *cubed complex* is a CW-complex X such that each (closed) k-cell of X with $k \geq 0$ is a continuous map from the k-dimensional cube I^k to X whose restriction to the interior of I^k is injective and whose restriction to each $(k-1)$-face of I^k is an isometry of that face onto I^{k-1} composed with a $(k-1)$-cell $I^{k-1} \to X$ of X. The k-cells $I^k \to X$ are not required to be injective. The k-skeleton X^k of X is the union of the images of all cells of dimension $\leq k$.

For example, the cube I^k together with all its faces is a cubed complex. The k-dimensional torus obtained by identifying opposite faces of I^k is a cubed complex.

The *link* $LK(A) = LK(A; X)$ of a 0-cell A of a cubed complex X is the space of all directions at A. Each triple ($k \geq 1$, a vertex a of I^k, a k-cell $\alpha : I^k \to X$ of X carrying a to A) determines a $(k-1)$-dimensional simplex in $LK(A)$ in the obvious way. The faces of this simplex are determined by the restrictions of α to the faces of I^k containing a. The simplices corresponding to all triples (k, a, α) cover $LK(A)$ but may not form a simplicial complex. We say, following [HaW], that the cubed complex X is *simple* if the links of all $A \in X^0$ are simplicial complexes, i.e., all simplices in $LK(A)$ are embedded and the intersection of any two simplices is a common face.

A *flag complex* is a simplicial complex such that any finite collection of pairwise adjacent vertices spans a simplex. A cubed complex is *nonpositively curved* if it is simple and the link of each 0-cell is a flag complex. A theorem of M. Gromov asserts that the universal covering of any connected finite-dimensional nonpositively curved cubed complex X is a CAT(0)-space. Since CAT(0)-spaces are contractible, all higher homotopy groups of such an X vanish while the fundamental group $\pi = \pi_1(X)$ is torsion-free. This group satisfies a strong form of the Tits alternative: each subgroup of π contains a rank 2 free subgroup or is virtually a finitely generated abelian group, see [SW]. Also, π does not have Kazhdan's property (T), see [NR1]. If X is compact, then π has solvable word and conjugacy problems and is biautomatic, see [NR2].

3.2. **Glide complexes.** Consider a group G endowed with a gliding system. We associate with G a cubed complex X_G called the *glide complex*.

We call a set of glides S *cubic* if S is finite, the glides in S are pairwise independent, and for any distinct subsets T_1, T_2 of S we have $[T_1] \neq [T_2]$. The following properties of cubic sets of glides are straightforward:

– any set of independent glides with ≤ 2 elements is cubic;

– all subsets of a cubic set of glides are cubic;

– for each subset T of a cubic set of glides S, the set $S_T = (S \setminus T) \cup \{t^{-1}\}_{t \in T}$ is a cubic set of glides.

A *based cube* in G is a pair ($A \in G$, a cubic set of glides $S \subset G$). Then, the set $\{[T]A\}_{T\subset S} \subset G$ has 2^k distinct elements where $k = \mathrm{card}(S)$ is the *dimension* of the based cube (A, S). These 2^k elements of G are called the *vertices* of (A, S).

Two based cubes (A, S) and (A', S') are *equivalent* if there is a set $T \subset S$ such that $A' = [T]A$ and $S' = S_T$ where S_T is defined above. This is indeed an equivalence relation on the set of based cubes. It is clear that each k-dimensional based cube is equivalent to 2^k based cubes (including itself). The equivalence classes of k-dimensional based cubes are called *k-dimensional cubes* or *k-cubes* in G. Since equivalent based cubes have the same vertices, we may speak of the vertices of a cube. The 0-cubes in G are just the elements of G.

A cube Q in G is a *face* of a cube Q' in G if Q, Q' may be represented by based cubes (A, S), (A', S'), respectively, such that $A = A'$ and $S \subset S'$. Note that in the role of A one may take an arbitrary vertex of Q.

The glide complex X_G is the cubed complex obtained by taking a copy of I^k for each k-cube in G with $k \geq 0$ and gluing these copies via identifications determined by inclusions of cubes into bigger cubes as their faces. Here is a more precise definition. A point of X_G is represented by a triple (A, S, x) where (A, S) is a based cube in G and x is a map $S \to I$. For fixed (A, S), such maps x form a geometric cube, a product of $\mathrm{card}(S)$ copies of I. We take a disjoint union of these cubes over all (A, S) and factorize it by the equivalence relation generated by the following relation: $(A, S, x) \sim (A', S', x')$ when $A = A', S \subset S', x = x'|_S, x'(S' \setminus S) = 0$ or there is a set $T \subset S$ such that $A' = [T]A, S' = S_T$, $x' = x$ on $S \setminus T$, and $x'(t^{-1}) = 1 - x(t)$ for all $t \in T$. The quotient space X_G is a cubed space in the obvious way. It is easy to see that all cubes forming X_G are embedded in X_G.

The formula $(A, S, x)g = (Ag, S, x)$ for $g \in G$ defines a right action of G on X_G preserving the cubed structure and free on the 0-skeleton $X^0_G = G$. When G has no elements of finite order, the action of G on X_G is free. This follows from the fact that the set of vertices of the minimal cube containing a fixed point of the action of $g \in G$ must be invariant under the action of g.

We further associate with any set $\mathcal{E} \subset G$ the cubed complex $X_{\mathcal{E}} \subset X_G$ formed by the cubes in G whose vertices lie in $\mathcal{E}$. Such cubes are said to be *cubes in $\mathcal{E}$*. The set of connected components of $X_{\mathcal{E}}$ is the quotient of $\mathcal{E}$ by the equivalence relation generated by glidings in $\mathcal{E}$. We call $X_{\mathcal{E}}$ the *glide complex* of $\mathcal{E}$. The fundamental groups of the components of $X_{\mathcal{E}}$ are called the *glide groups of $\mathcal{E}$*. Thus, each $A \in \mathcal{E}$ gives rise to a glide group $\pi_1(X_{\mathcal{E}}, A)$, and elements of $\mathcal{E}$ related by glidings in $\mathcal{E}$ give rise to isomorphic groups. If $\mathcal{E}$ is finite, then $X_{\mathcal{E}}$ is a finite CW-space and the glide groups are finitely presented.

Lemma 3.1. *The cubed complex $X_{\mathcal{E}}$ is simple for any $\mathcal{E} \subset G$.*

Proof. A neighborhood of $A \in G$ in X_G can be obtained by taking all triples (A, S, x), where S is a cubic set of glides and $x(S) \subset [0, 1/2)$, and identifying two such triples (A, S_1, x_1), (A, S_2, x_2) whenever $x_1 = x_2$ on $S_1 \cap S_2$ and $x_1(S_1 \setminus S_2) = x_2(S_2 \setminus S_1) = 0$. Therefore, the link $LK_G(A)$ of A in X_G is the simplicial complex whose vertices are glides and whose simplices are cubic sets of glides. The link $LK_{\mathcal{E}}(A)$ of A in $X_{\mathcal{E}}$ is the subcomplex of $LK_G(A)$ formed by the glides $s \in G$ such that $sA \in \mathcal{E}$ and the cubic sets S such that $[T]A \in \mathcal{E}$ for all $T \subset S$. $\square$

For $A \in \mathcal{E}$, we can reformulate the flag condition on $LK_{\mathcal{E}}(A)$ in terms of glides. Observe that the sets of pairwise adjacent vertices in $LK_{\mathcal{E}}(A)$ bijectively correspond to the sets of independent glides $S \subset G$ satisfying the following condition:

$(*)$ $sA \in \mathcal{E}$ for all $s \in S$ and $stA \in \mathcal{E}$ for all distinct $s, t \in S$.

Lemma 3.2. *$LK_{\mathcal{E}}(A)$ is a flag complex if and only if any finite set of independent glides $S \subset G$ satisfying $(*)$ is cubic and $[S]A \in \mathcal{E}$.*

This lemma follows directly from the definitions. One should use the obvious fact that if a set of independent glides satisfies $(*)$ then so do all its subsets.

We call a set $\mathcal{E} \subset G$ *regular* if for every $A \in \mathcal{E}$, all finite sets of independent glides $S \subset G$ satisfying $(*)$ are cubic.

Theorem 3.3. *For a group G with a gliding system and a set $\mathcal{E} \subset G$, the cubed complex $X_{\mathcal{E}}$ is nonpositively curved if and only if $\mathcal{E}$ is regular and meets the following 3-cube condition: if $A \in \mathcal{E}$ and pairwise independent glides $s_1, s_2, s_3 \in G$ satisfy $s_1A, s_2A, s_3A, s_1s_2A, s_1s_3A, s_2s_3A \in \mathcal{E}$, then $s_1s_2s_3A \in \mathcal{E}$.*

The 3-cube condition on $\mathcal{E}$ may be reformulated by saying that if seven vertices of a 3-cube in G belong to $\mathcal{E}$, then so does the eighth vertex. This condition is always fulfilled when $\mathcal{E} = G$ or, more generally, when $\mathcal{E}$ is a subgroup of G.

Proof. Lemmas 3.1 and 3.2 imply that $X_{\mathcal{E}}$ is nonpositively curved if and only if for every $A \in \mathcal{E}$, each finite set of independent glides S satisfying $(*)$ is cubic and $[S]A \in \mathcal{E}$. We must only show that the inclusion $[S]A \in \mathcal{E}$ can be replaced with the 3-cube condition. One direction is obvious: if $A, s_1, s_2, s_3 \in G$ satisfy the assumptions of the 3-cube condition, then the set $S = \{s_1, s_2, s_3\}$ satisfies $(*)$ and so $s_1s_2s_3A = [S]A \in \mathcal{E}$. Conversely, suppose that $\mathcal{E}$ is regular and meets the 3-cube condition. We must show that $[S]A \in \mathcal{E}$ for any $A \in \mathcal{E}$ and any finite set of independent glides $S \subset G$ satisfying $(*)$. We proceed by induction on $k = \mathrm{card}(S)$. For $k = 0$, the claim follows from the inclusion $A \in \mathcal{E}$. For $k = 1, 2$, the claim follows from $(*)$. If $k \geq 3$, then the induction assumption guarantees that $[T]A \in \mathcal{E}$ for any proper subset $T \subset S$. If $S = \{s_1, ..., s_k\}$, then applying the 3-cube condition to s_1, s_2, s_3 and the element $s_4 \cdots s_k A$ of $\mathcal{E}$, we obtain that $[S]A \in \mathcal{E}$. □

3.3. **Regular gliding systems.** A gliding system in a group G is *regular* if all finite sets of independent glides in G are cubic. This condition implies that all subsets of G are regular. Though we treat arbitrary gliding systems, we are mainly interested in regular ones. Without much loss, the reader may focus on the regular gliding systems.

Corollary 3.4. *For a group G with a regular gliding system, the glide complex of a set $\mathcal{E} \subset G$ is nonpositively curved if and only if $\mathcal{E}$ satisfies the 3-cube condition.*

By Section 3.1, if $X_{\mathcal{E}}$ is nonpositively curved, then the universal covering of every finite-dimensional component of $X_{\mathcal{E}}$ is a CAT(0)-space. The component itself is then an Eilenberg-MacLane space of type $K(\pi, 1)$ where π is the corresponding glide group. Note that $\dim X_{\mathcal{E}}$ is the maximal dimension of a cube in $\mathcal{E}$.

Corollary 3.5. *For a group G with a regular gliding system, the glide complex X_G is nonpositively curved.*

3.4. **Examples.** It is easy to see that the gliding systems in Examples 2.2.2–6 are regular. Corollaries 3.4 and 3.5 fully apply to these systems. The gliding system in Example 2.2.1, generally speaking, is not regular.

Here are a few further remarks on these examples.

In (2.2.2), X_G is the complete graph with the set of vertices G. The action of G on X_G is free if and only if G has no elements of order 2. All subsets of G satisfy the 3-cube condition and the corresponding glide groups are free.

In (2.2.3), if the rank, n, of G is finite, then $X_G = \mathbb{R}^n$.

In (2.2.4), X_G is simply-connected, the action of G on X_G is free, and the projection $X_G \to X_G/G$ is the universal covering of X_G/G. Composing the cells of X_G with this projection we turn X_G/G into a cubed complex called the *Salvetti complex*, see [Ch]. This complex has only one 0-cell whose link is isomorphic to the link of any vertex of X_G and is a flag complex. This recovers the well known fact that the Salvetti complex is nonpositively curved. Clearly, $\dim X_G = \dim(X_G/G)$ is the maximal number of vertices of a complete subgraph of Γ.

In (2.2.5), all elements of G have order 2 and the action of G on X_G is not free. If the set E is finite in (2.2.5), (2.2.6), then X_G is finite dimensional.

In Example 2.2.3 with $n \geq 3$ and in Examples 2.2.4–6, the 3-cube condition holds for some but not all $\mathcal{E} \subset G$.

4. Inclusion homomorphisms

In the setting of Section 3.2, a subset $\mathcal{F}$ of $\mathcal{E}$ determines a subcomplex $X_{\mathcal{F}}$ of $X_{\mathcal{E}}$. We formulate conditions ensuring that the inclusion $X_{\mathcal{F}} \hookrightarrow X_{\mathcal{E}}$ induces an injection of the fundamental groups.

4.1. **The square condition.** Let G be a group with a gliding system and $\mathcal{E} \subset G$. We say that a set $\mathcal{F} \subset \mathcal{E}$ satisfies the *square condition* $\operatorname{rel} \mathcal{E}$, if for any $A \in \mathcal{F}$ and any independent glides $s, t \in G$ such that $sA, tA \in \mathcal{F}$, $stA \in \mathcal{E}$, we necessarily have $stA \in \mathcal{F}$. For example, the intersection of $\mathcal{E}$ with any subgroup of G satisfies the square condition $\operatorname{rel} \mathcal{E}$.

Theorem 4.1. *Let $\mathcal{E} \subset G$ be a regular set satisfying the 3-cube condition and such that $\dim X_{\mathcal{E}} < \infty$. Let $\mathcal{F}$ be a subset of $\mathcal{E}$ satisfying the square condition $\operatorname{rel} \mathcal{E}$. Then the inclusion homomorphism $\pi_1(X_{\mathcal{F}}, A) \to \pi_1(X_{\mathcal{E}}, A)$ is injective for all $A \in \mathcal{F}$.*

Theorem 4.1 is proven in Section 4.3 using material of Section 4.2.

We say that a set $\mathcal{E} \subset G$ satisfies the *square condition* if it satisfies the square condition $\operatorname{rel} G$ that is for any $A \in \mathcal{E}$ and any independent glides $s, t \in G$ such that $sA, tA \in \mathcal{E}$, we necessarily have $stA \in \mathcal{E}$.

Corollary 4.2. *If $\dim X_G < \infty$, then for every regular set $\mathcal{E} \subset G$ satisfying the square condition and every $A \in \mathcal{E}$, the inclusion homomorphism $\pi_1(X_{\mathcal{E}}, A) \to \pi_1(X_G, A)$ is injective.*

It would be interesting to find out whether the conditions $\dim X_{\mathcal{E}} < \infty$ and $\dim X_G < \infty$ in these statements are necessary.

4.2. **Cubical maps.** Let X and Y be simple cubed complexes. A *cubical map* $f : X \to Y$ is a continuous map whose composition with any k-cell $I^k \to X$ of X splits as a composition of a self-isometry of I^k with a k-cell $I^k \to Y$ of Y for all $k \geq 0$. For every $A \in X^0$, such a map f induces a simplicial map

$f_A : LK(A) \to LK(f(A))$. The cubical map f is a *local isometry* if for all $A \in X^0$, the map f_A is an embedding onto a full subcomplex of $LK(f(A))$. Recall that a simplicial subcomplex Z' of a simplicial complex Z is *full* if any simplex of Z with vertices in Z' entirely lies in Z'.

Lemma 4.3. [CW, Theorem 1.2] *If $f : X \to Y$ is a local isometry of nonpositively curved finite-dimensional cubed complexes, then the induced homomorphism $f_* : \pi_1(X, A) \to \pi_1(Y, f(A))$ is injective for all $A \in X$.*

4.3. **Proof of Theorem 4.1.** Since $\mathcal{E}$ is regular, so is $\mathcal{F} \subset \mathcal{E}$. The 3-cube condition on $\mathcal{E}$ and the square condition on $\mathcal{F}$ imply that $\mathcal{F}$ satisifes the 3-cube condition. By Theorem 3.3, $X_\mathcal{E}$ and $X_\mathcal{F}$ are nonpositively curved. By assumption, the cubed complex $X_\mathcal{E}$ is finite-dimensional and so is its subcomplex $X_\mathcal{F}$. We claim that the inclusion $X_\mathcal{F} \hookrightarrow X_\mathcal{E}$ is a local isometry. Together with Lemma 4.3 this will imply the theorem.

To prove our claim, pick any $A \in \mathcal{F}$ and consider the simplicial complexes $L' = LK(A, X_\mathcal{F})$, $L = LK(A, X_\mathcal{E})$. The inclusion $X_\mathcal{F} \hookrightarrow X_\mathcal{E}$ induces an embedding $L' \hookrightarrow L$, and we need only to verify that the image of L' is a full subcomplex of L. Since L' is a flag simplicial complex, it suffices to verify that any vertices of L' adjacent in L are adjacent in L'. This follows from the square condition on $\mathcal{F}$.

5. Typing homomorphisms

5.1. **Artin groups of glides.** A group G carrying a gliding system $(\mathcal{G}, \mathcal{I})$ determines a right-angled Artin group $\mathcal{A} = \mathcal{A}(G)$ with generators $\{g_s\}_{s\in\mathcal{G}}$ and relations $g_s g_t g_s^{-1} g_t^{-1} = 1$ when $(s,t) \in \mathcal{I}$. This group is associated with the graph whose vertices are the glides in G and whose edges connect independent glides. By Example 2.2.4, the group $\mathcal{A}$ carries a gliding system with glides $\{g_s^{\pm 1}\}_{s\in\mathcal{G}}$. Two glides $g_s^{\pm 1}$ and $g_t^{\pm 1}$ in $\mathcal{A}$ are independent if and only if $s \neq t$ and $(s,t) \in \mathcal{I}$.

We shall relate the group $\mathcal{A}$ to the glide groups associated with a set $\mathcal{E} \subset G$. To this end, we introduce a notion of an orientation on $\mathcal{E}$. An *orientation* on $\mathcal{E}$ is a choice of direction on each 1-cell of the CW-complex $X_\mathcal{E}$ such that the opposite sides of any (square) 2-cell of $X_\mathcal{E}$ point towards each other on the boundary loop. This means that for any based square $(A, \{s,t\})$ with $A, sA, tA, stA \in \mathcal{E}$, the 1-cells connecting A to sA and tA to stA are directed either towards sA, stA or towards A, tA (and similarly with s and t exchanged). A set $\mathcal{E}$ is *orientable* if it has at least one orientation and is *oriented* if it has a distinguished orientation. An orientation of $\mathcal{E}$ induces an orientation of any subset $\mathcal{F} \subset \mathcal{E}$ via the inclusion $X_\mathcal{F} \subset X_\mathcal{E}$. Therefore, all subsets of an orientable set are orientable. These definitions apply, in particular, to $\mathcal{E} = G$. Examples of oriented sets will be given in Section 7.

Let $\mathcal{E} \subset G$ be an oriented set. We assign to each 1-cell e of $X_\mathcal{E}$ the glide $|e| = BA^{-1}$ where $A \in \mathcal{E}$ is the initial endpoint of e and $B \in \mathcal{E}$ is the terminal endpoint of e with respect to the distinguished orientation. Consider a path α in $X_\mathcal{E}$ joining two 0-cells and formed by $n \geq 0$ consecutive 1-cells $e_1, ..., e_n$ oriented so that the terminal endpoint of e_k is the initial endpoint of e_{k+1} for $k = 1, ..., n-1$. The orientation of e_k may coincide or not with that given by the orientation of $\mathcal{E}$. We set $\nu_k = +1$ or $\nu_k = -1$, respectively. Set

$$\mu(\alpha) = g_{|e_1|}^{\nu_1} g_{|e_2|}^{\nu_2} \cdots g_{|e_n|}^{\nu_n} \in \mathcal{A} = \mathcal{A}(G). \tag{5.1.1}$$

It is clear that $\mu(\alpha)$ is preserved under inserting in the sequence $e_1, ..., e_n$ two opposite 1-cells or four 1-cells forming the boundary of a 2-cell. Therefore $\mu(\alpha)$ is

preserved under homotopies of α in $X_{\mathcal{E}}$ relative to the endpoints. Applying μ to loops based at $A \in \mathcal{E}$, we obtain a homomorphism $\mu_A : \pi_1(X_{\mathcal{E}}, A) \to \mathcal{A}$. Following the terminology of [HaW], we call μ_A the *typing homomorphism*.

Theorem 5.1. *If there is an upper bound on the number of pairwise independent glides in G, then for any oriented regular set $\mathcal{E} \subset G$ satisfying the square condition and any $A \in \mathcal{E}$, the typing homomorphism $\mu_A : \pi_1(X_{\mathcal{E}}, A) \to \mathcal{A}$ is an injection.*

Proof. Consider the cubed complex $X = X_{\mathcal{A}}$ associated with the gliding system in $\mathcal{A}$ and the Salvetti complex $Y = X/\mathcal{A}$. Recall that $\mathcal{A} = \pi_1(Y, *)$ where $*$ is the unique 0-cell of Y. As explained in Section 3.4, both X and Y are nonpositively curved. The assumptions of the theorem imply that the cubed complex $X_{\mathcal{E}}$ is nonpositively curved, and the spaces X, Y, $X_{\mathcal{E}}$ are finite-dimensional. We claim that the homomorphism $\mu_A : \pi_1(X_{\mathcal{E}}, A) \to \mathcal{A} = \pi_1(Y, *)$ is induced by a local isometry $X_{\mathcal{E}} \to Y$. Together with Lemma 4.3 this will imply the theorem.

Since the gliding system in $\mathcal{A}$ is regular, the points of X are represented by triples $(A \in \mathcal{A}, \mathcal{S}, x : \mathcal{S} \to I)$ where $\mathcal{S}$ is a finite set of independent glides in $\mathcal{A}$ that is $S = \{g_s^{\varepsilon_s}\}_s$ where s runs over a finite set of independent glides in G and $\varepsilon_s \in \{\pm 1\}$. The space X is obtained by factorizing the set of such triples by the equivalence relation defined in Section 3.2. The space Y is obtained from X by forgetting the first term, A, of the triple. A point of Y is represented by a pair (a finite set of independent glides $\mathcal{S} \subset \mathcal{A}$, a map $x : \mathcal{S} \to I$). The space Y is obtained by factorizing the set of such pairs by the equivalence relation generated by the following relation: $(\mathcal{S}, x) \sim (\mathcal{S}', x')$ when $\mathcal{S} \subset \mathcal{S}', x = x'|_{\mathcal{S}}, x'(\mathcal{S}' \setminus \mathcal{S}) = 0$ or there is $T \subset \mathcal{S}$ such that $\mathcal{S}' = \mathcal{S}_T$, $x' = x$ on $\mathcal{S} \setminus T$, and $x'(t^{-1}) = 1 - x(t)$ for all $t \in T$.

We now construct a cubical map $f : X_{\mathcal{E}} \to Y$. The idea is to map each 1-cell, e, of $X_{\mathcal{E}}$ onto the 1-cell of Y determined by $g_{|e|}$ where $|e| \in G$ is the glide determined by the distinguished orientation of e as above. Here is a detailed definition of f. A point $a \in X_{\mathcal{E}}$ is represented by a triple $(A \in \mathcal{E}, S, x : S \to I)$ where S is a cubic set of glides in G such that $[T]A \in \mathcal{E}$ for all $T \subset S$. For $s \in S$, set $|s| = |e_s|$ where e_s is the 1-cell of $X_{\mathcal{E}}$ connecting A and sA. By definition, $|s| = s^{\varepsilon_s}$ where $\varepsilon_s = +1$ if e_s is oriented towards sA and $\varepsilon_s = -1$ otherwise. Let $f(a) \in Y$ be the point represented by the pair $(\mathcal{S} = \{g_{|s|}^{\varepsilon_s}\}_{s \in S}, y : \mathcal{S} \to I)$ where $y(g_{|s|}^{\varepsilon_s}) = x(s)$ for all $s \in S$. This yields a well-defined cubical map $f : X_{\mathcal{E}} \to Y$ inducing μ_A in π_1.

It remains to show that f is a local isometry. The link $L_A = LK_{\mathcal{E}}(A)$ of $A \in \mathcal{E}$ in $X_{\mathcal{E}}$ was described as a simplicial complex in the proof of Lemma 3.1. It has a vertex v_s for every glide $s \in G$ such that $sA \in \mathcal{E}$. A set of vertices $\{v_s\}_s$ spans a simplex in L_A whenever s runs over a cubic set of glides. The link, L, of $*$ in Y has two vertices w_s^+ and w_s^- for every glide $s \in G$. A set of vertices $\{w_s^{\pm}\}_s$ spans a simplex in L whenever s runs over a finite set of independent glides. The map $f_A : L_A \to L$ induced by f carries v_s to $w_{|s|}^{\varepsilon_s}$. This map is an embedding since we can recover s from $|s|$ and ε_s. The square condition on $\mathcal{E}$ implies that any vertices of L_A adjacent in L are adjacent in L_A. Since L_A is a flag simplicial complex, $f_A(L_A)$ is a full subcomplex of L. □

Corollary 5.2. *If the gliding system in G is regular, the number of pairwise independent glides in G is bounded from above, and G is orientable in the sense of Section 5.1, then $\mu_A : \pi_1(X_G, A) \to \mathcal{A}$ is an injection for all $A \in G$.*

This follows from Theorem 5.1 because the square condition on G is void.

5.2. **Applications.** Finitely generated right-angled Artin groups are biorderable, residually nilpotent, and embed in $SL_n(\mathbb{Z})$ for some n (so are residually finite), see [DuT], [DaJ], [CW], [HsW]. These properties are hereditary and are shared by all subgroups of finitely generated right-angled Artin groups. Combining with Theorem 5.1 we obtain the following.

Corollary 5.3. *If the set of glides in G is finite and an orientable regular set $\mathcal{E} \subset G$ satisfies the square condition, then for all $A \in \mathcal{E}$, the group $\pi_1(X_{\mathcal{E}}, A)$ is biorderable, residually nilpotent, and embeds in $SL_n(\mathbb{Z})$ for some n.*

5.3. **Remark.** It would be interesting to deduce Theorem 5.1 from the results of [HaW]. Are the glide complexes A-special in the sense of [HaW]?

6. Presentations of glide groups

Throughout this section, we fix a group G carrying a gliding system and a set $\mathcal{E} \subset G$. Under certain assumptions, we obtain a presentation of the glide group of $\mathcal{E}$ by generators and relations.

6.1. **Hulls.** Given a set $J \subset \mathcal{E}$ and a cube Q in $\mathcal{E}$ whose set of vertices contains J, we call Q the *hull of J in $\mathcal{E}$* if Q is a face of all cubes in $\mathcal{E}$ whose sets of vertices contain J. We refer to Section 3.2 for the definitions of cubes in $\mathcal{E}$, faces, etc. If a hull of J exists, then it is unique. If J has only one element, then J is a 0-cube and is its own hull.

Theorem 6.1. *If each 2-element subset of $\mathcal{E}$ has a hull in $\mathcal{E}$, then $X_{\mathcal{E}}$ is connected and for each $A_0 \in \mathcal{E}$, the group $\pi_1(X_{\mathcal{E}}, A_0)$ is canonically isomorphic to the group with generators $\{y_{A,B}\}_{A,B\in\mathcal{E}}$ subject to the following relations: $y_{A,C} = y_{A,B}\, y_{B,C}$ for every triple A, B, C of vertices of a cube in $\mathcal{E}$ and $y_{A_0,A} = 1$ for all $A \in \mathcal{E}$. The image of each $y_{A,B}$ in $\pi_1(X_{\mathcal{E}}, A_0)$ is represented by the loop in $X_{\mathcal{E}}$ composed of a path from A_0 to A in the hull of $\{A_0, A\}$, a path from A to B in the hull of $\{A, B\}$, and a path from B to A_0 in the hull of $\{A_0, B\}$.*

Theorem 6.1 is proved in Section 6.4 using Sections 6.2 and 6.3 which are concerned with more general situations. One can check that $A, B, C \in \mathcal{E}$ are vertices of a certain cube in $\mathcal{E}$ if and only if there is $K \in \mathcal{E}$ and a cubic set of glides $S \subset G$ such that $[T]K \in \mathcal{E}$ for all $T \subset S$ and S has a partition $S = X \amalg Y \amalg Z$ such that $A = [X]K$, $B = [Y]K$, and $C = [Z]K$.

6.2. **The group F.** Let F be the group generated by symbols labeling 1-cells of $X_{\mathcal{E}}$ subject to the relations associated with 2-cells of $X_{\mathcal{E}}$. More precisely, F is generated by the set

$$\{x_{A,B} \,|\, A, B \in \mathcal{E} \text{ such that } AB^{-1} \text{ is a glide}\}.$$

subject to the relations $x_{A,B}\, x_{B,A} = 1$ for any A, B and

$$x_{A,B}\, x_{B,C}\, x_{C,D}\, x_{D,A} = 1 \tag{6.2.1}$$

for any $A, B, C, D \in \mathcal{E}$ such that $B = sA, C = stA, D = tA$ for some independent glides $s, t \in G$. For a path α in $X_{\mathcal{E}}$ formed by 1-cells and consecutively connecting 0-cells A_0, A_1,..., $A_n \in \mathcal{E}$, set

$$\phi(\alpha) = \prod_{k=1}^{n} x_{A_{k-1},A_k} \in F. \tag{6.2.2}$$

It is clear that $\phi(\alpha)$ is preserved under homotopies of α relative to the endpoints.

Lemma 6.2. *Applying ϕ to loops based at $A_0 \in \mathcal{E}$, we obtain a homomorphism $\phi : \pi_1(X_\mathcal{E}, A_0) \to F$. This homomorphism is a split injection.*

Proof. The first claim is obvious. To prove the second claim, denote by X the component of $X_\mathcal{E}$ containing A_0. For each 0-cell $A \in X$ pick a path ρ_A from A_0 to A in X. For ρ_{A_0} we take the constant path at A_0. Given 0-cells $A, B \in X$ such that AB^{-1} is a glide, denote by $\overline{AB}$ the path from A to B running along the unique 1-cell of X connecting A and B. Set

$$\psi(x_{A,B}) = [\rho_A \overline{AB} \rho_B^{-1}] \in \pi_1(X, A_0) = \pi_1(X_\mathcal{E}, A_0)$$

where the square brackets stand for the homotopy class of a loop. For 0-cells A, B not lying in X, set $\psi(x_{A,B}) = 1$. The formula $x_{A,B} \mapsto \psi(x_{A,B})$ is compatible with the defining relations of F and yields a homomorphism $\psi : F \to \pi_1(X_\mathcal{E}, A_0)$. It follows from the definitions that $\psi\phi = \mathrm{id}$. □

Though we shall not need it, note that for oriented $\mathcal{E}$, there is a homomorphism $g : F \to \mathcal{A}(G)$ carrying $x_{A,B}$ and $x_{B,A}$ respectively to $g_{AB^{-1}}$ and $(g_{AB^{-1}})^{-1}$ when there is a 1-cell of $X_\mathcal{E}$ oriented from A to B. Then $g\phi = \mu_{A_0} : \pi_1(X_\mathcal{E}, A_0) \to \mathcal{A}(G)$.

6.3. **Central points.** We call $A_0 \in \mathcal{E}$ *central* if for any $A \in \mathcal{E}$, the set $\{A_0, A\}$ has a hull in $\mathcal{E}$. This means that there is a cubic set of glides $S \subset G$ such that $A \in \{[T]A_0\}_{T \subset S} \subset \mathcal{E}$ and that any cubic set of glides with these properties contains S. The vertices A, A_0 of the based cube (A_0, S) form then a big diagonal and so $A = [S]A_0$. The vertices $\{s^{-1}A\}_{s \in S}$ of this cube are called the *ascendents* of A. The set of ascendents of A is empty if and only if $A = A_0$.

Lemma 6.3. *If $A_0 \in \mathcal{E}$ is central, then $X_\mathcal{E}$ is connected and the group $\pi_1(X_\mathcal{E}, A_0)$ is isomorphic to F/N where F is the group defined in Section 6.2 and N is the normal subgroup of F generated by the set $\{x_{A,B}\}$ where A, B run over all elements of $\mathcal{E}$ such that A is an ascendent of B.*

Proof. Pick $A \in \mathcal{E}$ and represent the hull of $\{A_0, A\}$ by a based cube (A_0, S) where $S = \{s_1, ..., s_n\}$ is a cubic set of glides with $n \geq 0$. As we know, $A = [S]A_0 = s_1 \cdots s_n A_0$. For $k = 1, ..., n$, set $A_k = s_1 \cdots s_k A_0$. The cube in $\mathcal{E}$ represented by the based cube $(A_0, \{s_1, ..., s_k\})$ contains A_0, A_k among its vertices and has no proper faces with the same property. Therefore this cube is the hull of $\{A_0, A_k\}$ in $\mathcal{E}$. Hence, A_{k-1} is an ascendent of A_k for all k.

The edges of $X_\mathcal{E}$ connecting consecutively $A_0, A_1, ..., A_n = A$ form a path ρ_A from A_0 to A. The homotopy class of ρ_A in $X_\mathcal{E}$, obviously, does not depend on the order in S. Recall the homomorphism $\phi : \pi_1(X_\mathcal{E}, A_0) \to F$ from Lemma 6.2. By definition, $\phi(\rho_A) = x_{A_0,A_1} x_{A_1,A_2} \cdots x_{A_{n-1},A_n}$. All generators on the right-hand side belong to N because A_{k-1} is an ascendent of A_k for all k. Hence, $\phi(\rho_A) \in N$ for all $A \in \mathcal{E}$. Note that ρ_{A_0} is the constant path at A_0.

The existence of the paths $\{\rho_A\}_A$ implies that $X_\mathcal{E}$ is connected. As in Lemma 6.2, these paths determine a homomorphism $\psi : F \to \pi_1(X_\mathcal{E}, A_0)$ by $\psi(x_{A,B}) = [\rho_A \overline{AB} \rho_B^{-1}]$. Observe that $N \subset \operatorname{Ker} \psi$. Indeed, for any $A, B \in \mathcal{E}$ such that A is an ascendent of B, we have $\rho_B = \rho_A \overline{AB}$ and

$$\psi(x_{A,B}) = [\rho_A \overline{AB} \rho_B^{-1}] = [\rho_A \overline{AB}\, \overline{AB}^{-1} \rho_A^{-1}] = 1.$$

We claim that the homomorphisms $\tilde{\phi} : \pi_1(X_{\mathcal{E}}, A_0) \to F/N$ and $\tilde{\psi} : F/N \to \pi_1(X_{\mathcal{E}}, A_0)$ induced by ϕ and ψ respectively, are mutually inverse isomorphisms. Clearly, $\tilde{\psi}\tilde{\phi} = \psi\phi = \mathrm{id}$. To check that $\tilde{\phi}\tilde{\psi} = \mathrm{id}$, pick any $A, B \in \mathcal{E}$ such that AB^{-1} is a glide. Denote the projection of $x_{A,B} \in F$ to F/N by $\tilde{x}_{A,B}$. We have

$$\tilde{\phi}\tilde{\psi}(\tilde{x}_{A,B}) = \tilde{\phi}\psi(x_{A,B}) = \tilde{\phi}([\rho_A \overline{AB} \rho_B^{-1}]) = \tilde{\phi}(\rho_A)\, \tilde{x}_{A,B}\, \tilde{\phi}(\rho_B)^{-1} = \tilde{x}_{A,B}$$

where at the last step we use that $\phi(\rho_A), \phi(\rho_B) \in N$. Therefore $\tilde{\phi}\tilde{\psi} = \mathrm{id}$. □

6.4. **Proof of Theorem 6.1.** Let Π be the group defined by the generators and relations in this theorem. The relation $y_{A_0,A}\, y_{A,A} = y_{A_0,A}$ implies that $y_{A,A} = 1$ for all $A \in \mathcal{E}$. Then $y_{A,B}\, y_{B,A} = y_{A,A} = 1$ for all $A, B \in \mathcal{E}$ so that $y_{B,A} = y_{A,B}^{-1}$. In particular, $y_{A,A_0} = y_{A_0,A}^{-1} = 1$ for all $A \in \mathcal{E}$.

The assumptions of Theorem 6.1 imply that $A_0 \in \mathcal{E}$ is central. By Lemma 6.3, to prove the first part of the theorem, it suffices to construct an isomorphism $\beta : F/N \to \Pi$. We define β on the generators by $\beta(x_{A,B}) = y_{A,B}$. The compatibility with the relation $x_{A,B}\, x_{B,A} = 1$ in F is clear. The compatibility with the relation (6.2.1) follows from the equalities

$$y_{A,B}\, y_{B,C}\, y_{C,D}\, y_{D,A} = y_{A,C}\, y_{C,D}\, y_{D,A} = y_{A,D}\, y_{D,A} = 1.$$

Finally, if $A \in \mathcal{E}$ is an ascendent of $B \in \mathcal{E}$, then A, B, A_0 lie in a cube in $\mathcal{E}$ and so $\beta(x_{A,B}) = y_{A,B} = y_{A,A_0}\, y_{A_0,B} = 1$.

Next, we construct a homomorphism $\gamma : \Pi \to F/N$. For $A, B \in \mathcal{E}$, let $Q = Q(A,B)$ be the hull of $\{A, B\}$ in $\mathcal{E}$. We can connect A to B by a path formed by edges of Q and visiting consecutively vertices $A = A_1, A_2, ..., A_n = B$ of Q. Any two such paths are related by homotopies pushing the path across square faces of Q. The relations in F ensure that $\gamma(y_{A,B}) = \prod_k x_{A_{k-1},A_k}$ is a well defined element of F/N. If $A, B, C \in \mathcal{E}$ are vertices of a cube in $\mathcal{E}$, then the hulls $Q(A,B), Q(A,C), Q(B,C)$ are faces of this cube, and an argument similar to the one above shows that $\gamma(y_{A,B}\, y_{B,C}) = \gamma(y_{A,C})$. Finally, a path connecting A_0 to A may be chosen so that each vertex is an ascendent of the next vertex. By definition of N, this gives $\gamma(y_{A_0,A}) = 1$.

For each generator $x_{A,B}$ of F/N, there is a 1-cell P of $X_{\mathcal{E}}$ connecting A to B. Clearly, P is a 1-cube in $\mathcal{E}$ with vertices A, B. No proper face of P contains both A and B. Hence, $Q(A,B) = P$ and $\gamma\beta(x_{A,B}) = \gamma(y_{A,B}) = x_{A,B}$. Therefore $\gamma\beta = \mathrm{id}$.

For any $A, B \in \mathcal{E}$, we have (in the notation above)

$$\beta\gamma(y_{A,B}) = \prod_k \beta(x_{A_{k-1},A_k}) = \prod_k y_{A_{k-1},A_k} = y_{A,B}$$

where the last equality follows from the defining relations in Π. Thus, $\beta\gamma = \mathrm{id}$.

The second claim of the theorem follows from the definitions of γ and $\tilde{\psi}$ above.

6.5. **The fundamental groupoid.** We now compute the fundamental groupoid $\pi = \pi_1(X_{\mathcal{E}}, \mathcal{E})$ of the pair $(X_{\mathcal{E}}, \mathcal{E})$. Recall that a groupoid is a (small) category in which all morphisms are isomorphisms. The groupoid π is the category whose objects are elements of $\mathcal{E}$ and whose morphisms are homotopy classes of paths in $X_{\mathcal{E}}$ with endpoints in $\mathcal{E}$. In the next assertion, we write down composition of morphisms in π in the order opposite to the usual one; this makes composition in π compatible with multiplication of paths.

Corollary 6.4. *Under the conditions of Theorem 6.1, the groupoid $\pi = \pi_1(X_{\mathcal{E}}, \mathcal{E})$ has generators $\{z_{A,B} : A \to B\}_{A,B\in\mathcal{E}}$ subject to the defining relations $z_{A,C} = z_{A,B}\, z_{B,C}$ for each triple A, B, C of vertices of a cube in $\mathcal{E}$. The image of each $z_{A,B}$ in $\pi_1(X_{\mathcal{E}}, \mathcal{E})$ is represented by a path from A to B in the hull of $\{A, B\}$.*

Proof. Let Z be the groupoid whose set of objects is $\mathcal{E}$ and whose morphisms are defined by the generators and relations in this corollary (and are invertible). It follows from the relations that $z_{A,A} = 1$ and $z^{-1}_{A,B} = z_{B,A}$ for all $A, B \in \mathcal{E}$. Consider the functor $f : Z \to \pi$ which is the identity on the objects and which carries each $z_{A,B}$ to the homotopy class of paths from A to B in the hull of $\{A, B\}$. Since each path between 0-cells of $X_{\mathcal{E}}$ is homotopic to a path in the 1-skeleton of $X_{\mathcal{E}}$, the functor f is surjective on the morphisms. To prove that f is injective it is enough to prove that for any $A_0 \in \mathcal{E}$, the map

$$f_{A_0} : \mathrm{End}_Z(A_0) \to \mathrm{End}_\pi(A_0) = \pi_1(X_{\mathcal{E}}, A_0)$$

induced by f is injective. Theorem 6.1 implies that the formula

$$g(y_{A,B}) = z_{A_0,A} z_{A,B} z_{B,A_0} \in \mathrm{End}_Z(A_0)$$

defines a homomorphism $g : \pi_1(X_{\mathcal{E}}, A_0) \to \mathrm{End}_Z(A_0)$. It is easy to check that $f_{A_0} g = \mathrm{id}$ and g is onto. Hence, g and f_{A_0} are bijections. □

7. Set-like glidings

We study a class of gliding systems generalizing Example 2.2.5.

7.1. **Set-like gliding systems.** Consider a set E and its power group $G = 2^E \cong (\mathbb{Z}/2\mathbb{Z})^E$. A gliding system in G is *set-like* if any independent glides in this system are disjoint as subsets of E. Thus, a set-like gliding system in G consists of a set $\mathcal{G} \subset G \setminus \{1\}$ and a symmetric set $\mathcal{I} \subset \mathcal{G} \times \mathcal{G}$ such that $(s, t) \in \mathcal{I} \Longrightarrow s \cap t = \emptyset$. For instance, the gliding system in Example 2.2.5 is set-like.

Lemma 7.1. *A set-like gliding system is regular in the sense of Section 3.3.*

Proof. We must show that any finite set of independent glides $S \subset G$ is cubic. Let T_1, T_2 be distinct subsets of S. Assume for concreteness that there is $t \in T_1 \setminus T_2$. Since all glides in S are disjoint as subsets of E, we have $t \subset [T_1]$ and $t \cap [T_2] = \emptyset$ where $[T_i] = \cup_{s\in T_i} s$ for $i = 1, 2$. Thus, $[T_1] \neq [T_2]$. Hence S is cubic. □

In the rest of this section we discuss other properties of set-like gliding systems. Fix from now on a set E and a set-like gliding system in $G = 2^E$.

7.2. **Orientation.** We claim that all subsets of G are orientable in the sense of Section 5.1. It suffices to orient the set G itself because an orientation on G induces an orientation on any subset of G. Pick an element $e_s \in s$ in every glide $s \subset E$. A 1-cell of X_G relates two 0-cells $A, B \subset E$ such that $AB = (A \setminus B) \cup (B \setminus A)$ is a glide in G. Then $e_{AB} \in AB$ belongs to precisely one of the sets A, B. We orient the 1-cell in question towards the 0-cell containing e_{AB}. It is easy to check that this defines an orientation on G. (Here we need to use the assumption that independent glides are disjoint.)

An orientation on G determines a homomorphism $\mu_A : \pi_1(X_G, A) \to \mathcal{A}(G)$ for all $A \in G$, see Section 5.1. If E is finite, then the right-angled Artin group $\mathcal{A}(G)$ is finitely generated, μ_A is injective (Corollary 5.2), and therefore the group $\pi_1(X_G, A)$ is biorderable, residually nilpotent, and embeds in $SL_n(\mathbb{Z})$ for some n.

7.3. **The evaluation map.** Let I^E be the set of maps $E \to I$ viewed as the product of copies of I numerated by elements of E and provided with the product topology. Assigning to each set $A \subset E$ its characteristic function $\delta_A : E \to \{0,1\} \subset I$ we obtain a map from the 0-skeleton of X_G to I^E. We extend this map to X_G as follows. For a point $a \in X_G$ represented by a triple $(A, S, x : S \to I)$ as in Section 3.2, let $\omega(a) : E \to I$ be the map defined by

$$\omega(a)(e) = \begin{cases} x(s) & \text{for } e \in s \setminus A \text{ with } s \in S \\ 1 - x(s) & \text{for } e \in s \cap A \text{ with } s \in S \\ \delta_A(e) & \text{for all other } e \in E\,. \end{cases} \tag{7.3.1}$$

This definition makes sense because distinct $s \in S$ are disjoint as subsets of E. The definition of $\omega(a)$ may be rewritten in terms of characteristic functions:

$$\omega(a) = \delta_A + (1 - 2\delta_A) \sum_{s \in S} x(s)\delta_s.$$

It is easy to check that the formula $a \mapsto \omega_a$ yields a well-defined continuous map $\omega : X_G \to I^E$ whose restriction to any cube in X_G is an embedding. We call ω the *evaluation map.*

Lemma 7.2. *Let $\mathcal{E} \subset G = 2^E$ and the following two conditions are met:*
(i) $[S_1] = [S_2] \Longrightarrow S_1 = S_2$ for any cubic sets of glides $S_1, S_2 \subset G$;
(ii) any glide $s \subset E$ has a partition into two non-empty subsets such that if $A \in \mathcal{E}$ and $sA \in \mathcal{E}$, then $s \cap A$ is one of these subsets.
Then the restriction of $\omega : X_G \to I^E$ to $X_{\mathcal{E}} \subset X_G$ is an injection.

Proof. Suppose that $\omega(a_1) = \omega(a_2)$ for some $a_1, a_2 \in X_{\mathcal{E}}$. Represent each a_i by a triple $(A_i \in \mathcal{E}, S_i, x_i : S_i \to I)$. Passing, if necessary, to a face of the cube, we can assume that $0 < x_i(s) < 1$ for all $s \in S_i$. Set $f_i = \omega(a_i) : E \to I$. Clearly, $f_i^{-1}((0,1)) = \cup_{s \in S_i} s = [S_i]$. The equality $f_1 = f_2$ implies that $[S_1] = [S_2]$. By condition (i), $S_1 = S_2$. Set $S = S_1 = S_2$ and $f = f_1 = f_2 : E \to I$.

We prove below that for any glide $s \in S$, either $(*)$ $s \cap A_1 = s \cap A_2$ and $x_1(s) = x_2(s)$ or $(**)$ $s \cap A_1 = s \setminus A_2$ and $x_1(s) = 1 - x_2(s)$. For each s of the second type, replace (A_2, S, x_2) with $(A_2' = sA_2, S, x_2')$ where $x_2' : S \to I$ carries s to $1 - x(s)$ and is equal to x_2 on $S \setminus \{s\}$. The triple (A_2', S, x_2') represents the same point $a_2 \in \mathcal{E}$ and satisfies $s \cap A_1 = s \cap A_2'$ and $x_1(s) = x_2'(s)$. Since different $s \in S$ are disjoint as subsets of E, such replacements along various s do not interfere with each other and commute. Effecting them for all s of the second type, we obtain a new triple (A_2, S, x_2) representing a_2 and satisfying $s \cap A_1 = s \cap A_2$, $x_1(s) = x_2(s)$ for all $s \in S$. Then $x_1 = x_2 : S \to I$. Also,

$$A_1 \cup \cup_{s \in S} s = f^{-1}((0,1]) = A_2 \cup \cup_{s \in S} s.$$

Since different $s \in S$ are disjoint as subsets of E and satisfy $s \cap A_1 = s \cap A_2$, we deduce that $A_1 = A_2$. Thus, $a_1 = a_2$.

It remains to prove that for any $s \in S$, we have either $(*)$ or $(**)$. By (7.3.1), the function $f : E \to I$ takes at most two values on $s \subset E$. Suppose first that $f|_s$ takes two distinct values (whose sum is equal to 1). By (7.3.1), f is constant on both $s \cap A_i$ and $s \setminus A_i$ for $i = 1, 2$. If $f(s \cap A_1) = f(s \cap A_2)$, then we have $(*)$. If $f(s \cap A_1) = f(s \setminus A_2)$, then we have $(**)$. Suppose next that f takes only one value on s. Condition (ii) implies that $s \cap A_1 \neq \emptyset$ and $s \setminus A_1 \neq \emptyset$. Since $f = \omega(a_1)$ takes only one value on s, formula (7.3.1) implies that this value is $1/2$.

Thus, $x_1(s) = x_2(s) = 1/2$. Applying condition (ii) again, we obtain that either $s \cap A_1 = s \cap A_2$ or $s \cap A_1 = s \setminus A_2$. This gives, respectively, $(*)$ or $(**)$. $\square$

8. The dimer complex

8.1. **Cycles in graphs.** By a *graph* we mean a 1-dimensional CW-complex without loops (= 1-cells with coinciding endpoints). The 0-cells and (closed) 1-cells of a graph are called *vertices* and *edges*, respectively. We allow multiple edges with the same vertices. A set s of edges of a graph Γ is *cyclic* if s is finite and each vertex of Γ either is incident to two distinct edges belonging to s or is not incident at all to edges belonging to s. Then, the union, $\underline{s}$, of the edges belonging to s is a subgraph of Γ homeomorphic to a disjoint union of a finite number of circles. A cyclic set of edges s is a *cycle* if $\underline{s}$ is a single circle. A cycle is *even* if it has an even number of elements. An even cycle s has a partition into two subsets called the *halves* obtained by following along the circle $\underline{s}$ and collecting all odd-numbered edges into one subset and all even-numbered edges into another subset. The edges belonging to the same half have no common endpoints.

8.2. **The gliding system.** Let Γ be a graph with the set of edges E and let $G = G(\Gamma) = 2^E$ be the power group of E. Two sets $s, t \subset E$ are *independent* if the edges belonging to s have no common vertices with the edges belonging to t. This condition implies that $s \cap t = \emptyset$.

Lemma 8.1. *The even cycles in Γ in the role of glides together with the independence relation above form a regular set-like gliding system in G.*

Proof. All conditions on a gliding system are straightforward. The resulting gliding system is obviously set-like in the sense of Section 7.1. It is regular because by Lemma 7.1, all set-like gliding systems are regular. $\square$

Recall that for a set $A \subset E$ and an even cycle s in Γ, the gliding of A along s gives $sA = (s \setminus A) \cup (A \setminus s)$. We view sA as the set obtained from A be removing the edges belonging to $s \cap A$ and adding all the other edges of s.

By Section 3.2, the gliding system of Lemma 8.1 determines a cubed complex X_G with 0-skeleton G. By Corollary 3.5, this complex is nonpositively curved.

8.3. **Dimer coverings.** Let Γ be a graph with the set of edges E. A *dimer covering*, or a *perfect matching*, on Γ is a subset of E such that every vertex of Γ is incident to exactly one edge of Γ belonging to this subset. Let $\mathcal{D} = \mathcal{D}(\Gamma) \subset G = 2^E$ be the set (possibly, empty) of all dimer coverings on Γ.

We provide the power group G with the gliding system of Lemma 8.1. By Section 3.2, the set $\mathcal{D} \subset G$ determines a cubed complex $X_{\mathcal{D}} \subset X_G$ with 0-skeleton $\mathcal{D}$. We call $X_{\mathcal{D}}$ the *dimer complex* of Γ. By definition, two dimer coverings $A, B \in \mathcal{D}$ are connected by an edge in $X_{\mathcal{D}}$ if and only if $AB \subset E$ is an even cycle. Note that if $s = AB$ is a cycle then it is necessarily even with complementary halves $s \cap A = A \setminus B$ and $s \cap B = B \setminus A$.

Lemma 8.2. *The set of dimer coverings $\mathcal{D} \subset G$ satisfies the 3-cube condition of Section 3.2 and the square condition of Section 4.1.*

Proof. The 3-cube condition follows from the square condition which says that for any $A \in \mathcal{D}$ and any independent even cycles s, t in Γ such that $sA, tA \in \mathcal{D}$, we have $stA \in \mathcal{D}$. The inclusions $A, sA \in \mathcal{D}$ imply that $s \cap A$ and $s \setminus A$ are the halves of s. A

similar claim holds for t. The independence of s, t means that s, t are disjoint and incident to disjoints sets of vertices. The set $stA \subset E$ is obtained from A through simultaneous replacement of the half $s \cap A$ of s and the half $t \cap A$ of t with the complementary halves. It is clear that stA is a dimer covering. □

Theorem 8.3. *The dimer complex $X_{\mathcal{D}}$ is nonpositively curved.*

This theorem follows from Lemma 8.2 and Corollary 3.4.

8.4. **Examples.** 1. Let Γ be a triangle (with 3 vertices and 3 edges). The set of glides in $G = G(\Gamma)$ is empty, $X_G = G$ consists of 8 points, and $X_{\mathcal{D}} = \mathcal{D} = \emptyset$.

2. Let Γ be a square (with 4 vertices and 4 edges). Then $G = G(\Gamma)$ has one glide, X_G is a disjoint union of 8 closed intervals, and $X_{\mathcal{D}}$ is one of them.

3. More generally, let Γ be formed by $n \geq 1$ cyclically connected vertices and $\mathcal{D} = \mathcal{D}(\Gamma)$. If n is odd, then $X_{\mathcal{D}} = \mathcal{D} = \emptyset$. If n is even, then Γ has two dimer coverings and $X_{\mathcal{D}}$ is a segment.

4. Let Γ be formed by 2 vertices and 3 connecting them edges. Then $G = G(\Gamma)$ has 3 glides and X_G is a disjoint union of two complete graphs on 4 vertices. The space $X_{\mathcal{D}}$ is formed by 3 vertices and 3 edges of one of these graphs.

5. More generally, for $n \geq 1$, consider the graph Γ^n formed by 2 vertices and n connecting them edges. A dimer covering of Γ^n consists of a single edge, and so, the set $\mathcal{D} = \mathcal{D}(\Gamma^n)$ has n elements. The graph Γ^n has $n(n-1)/2$ cycles, all of length 2 and none of them independent. The complex $X_{\mathcal{D}}$ is a complete graph on n vertices.

8.5. **Remark.** If Γ is a disjoint union of graphs Γ_1, Γ_2, then $G = G(\Gamma) = G_1 \times G_2$, where $G_i = G(\Gamma_i)$ for $i = 1, 2$, and $X_G = X_{G_1} \times X_{G_2}$. If the graphs Γ_1, Γ_2 are finite, then $\mathcal{D} = \mathcal{D}(\Gamma) = \mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_i = \mathcal{D}(\Gamma_i)$ for $i = 1, 2$, and $X_{\mathcal{D}} = X_{\mathcal{D}_1} \times X_{\mathcal{D}_2}$.

9. The dimer group

We focus now on the case of finite graphs, i.e., graphs with finite sets of edges and vertices. Throughout this section, Γ is a finite graph with set of edges E. Set $G = G(\Gamma) = 2^E$ and $\mathcal{D} = \mathcal{D}(\Gamma) \subset G$.

9.1. **The group $D(\Gamma)$.** The finiteness of Γ implies that the cubed complexes X_G and $X_{\mathcal{D}} \subset X_G$ are finite CW-spaces.

Lemma 9.1. *Every 2-element subset of $\mathcal{D}$ has a hull in $\mathcal{D}$ (cf. Section 6.1).*

Proof. Let $A, B \in \mathcal{D}$ be dimer coverings of Γ. Then $AB = (A \setminus B) \cup (B \setminus A)$ is a cyclic set of edges. It splits uniquely as a union of independent cycles. Denote the set of these cycles by S. All cycles in S are even: their halves are their intersections with $A \setminus B$ and $B \setminus A$. In the notation of Section 2.1, $[S] = AB$. As in the proof of Lemma 8.2, one easily shows that for all $T \subset S$, the set $[T]A \subset E$ is a dimer covering of Γ. The based cube (A, S) determines a cube, Q, in $\mathcal{D}$ whose set of vertices contains A and $B = [S]A$. An arbitrary cube Q' in $\mathcal{D}$ whose set of vertices contains A and B can be represented by a based cube (A, S') such that $B = [T]A$ for some $T \subset S'$. We have $[T] = BA^{-1} = AB$. Since a cyclic set of edges splits as a union of independent cycles in a unique way, $T = S$. So, Q is a face of Q'. □

Lemma 9.1 implies that $X_{\mathcal{D}}$ is connected, and Theorem 8.3 implies that $X_{\mathcal{D}}$ is aspherical. We call the fundamental group of $X_{\mathcal{D}}$ the *dimer group* of Γ and denote it $D(\Gamma)$. We refer to Section 3.1 for properties of $D(\Gamma)$ which follow from Theorem 8.3. If Γ does not have dimer coverings, then $X_{\mathcal{D}} = \emptyset$ and $D(\Gamma) = \{1\}$.

In Examples 8.4.1-3, $D(\Gamma) = \{1\}$. In 8.4.4, $D(\Gamma) = \mathbb{Z}$. In 8.4.5, the group $D(\Gamma^n)$ is a free group of rank $(n-1)(n-2)/2$. By Remark 8.5, if Γ is a disjoint union of finite graphs Γ_1, Γ_2, then $D(\Gamma) = D(\Gamma_1) \times D(\Gamma_2)$. Therefore any free abelian group of finite rank can be realized as the dimer group of a finite graph (see also Remark 9.5.1). Other abelian groups cannot be realized as dimer groups because all dimer groups are finitely generated and torsion-free.

Theorem 9.2. *The inclusion homomorphism $D(\Gamma) = \pi_1(X_{\mathcal{D}}, A) \to \pi_1(X_G, A)$ is injective for all $A \in \mathcal{D}$.*

Theorem 9.2 follows from Corollary 4.2 and Lemma 8.2.

9.2. **Typing homomorphisms.** By Section 7.2, a choice of an element $e_s \in s$ in each even cycle s in Γ determines an orientation on G which, in its turn, determines a homomorphism $\mu_A : \pi_1(X_G, A) \to \mathcal{A}$ for all $A \in G$. Here $\mathcal{A} = \mathcal{A}(G)$ is the right-angled Artin group associated with the gliding system in G. Since E is finite, the group $\mathcal{A}$ is finitely generated and μ_A is an injection. Composing μ_A with the inclusion homomorphism $\pi_1(X_{\mathcal{D}}, A) \to \pi_1(X_G, A)$ we obtain the *typing homomorphism* $\pi_1(X_{\mathcal{D}}, A) \to \mathcal{A}$ also denoted μ_A. The same homomorphism is obtained by restricting the orientation on G to $\mathcal{D}$ and considering the associated typing homomorphism as in Section 5.1. The homomorphism $\mu_A : \pi_1(X_{\mathcal{D}}, A) \to \mathcal{A}$ embeds the dimer group into a finitely generated right-handed Artin group. We refer to Section 5.2 for properties of the dimer group which follow from this fact.

Though we shall not need it in the sequel, we state here a few facts concerning the typing homomorphisms. First of all, two families $\{e_s \in s\}_s$ and $\{e'_s \in s\}_s$ determine the same orientation on $\mathcal{D}$ if and only if e_s, e'_s belong to the same half of s for each even cycle s in Γ. Thus, the orientations on $\mathcal{D}$ associated with such families are fully determined by a choice of a half in each s. If the distinguished half of a cycle s_0 is replaced with the complementary half, the typing homomorphism $\mu_A : \pi_1(X_{\mathcal{D}}, A) \to \mathcal{A}$ is replaced by its composition with the automorphism of $\mathcal{A}$ inverting the generator $g_{s_0} \in \mathcal{A}$ and fixing the generators $g_s \in \mathcal{A}$ for all $s \neq s_0$. Thus, μ_A does not depend on the choice of the halves up to composition with automorphisms of $\mathcal{A}$. Finally, we point out a non-trivial homomorphism u from $\mathcal{A}$ to another group such that $\operatorname{Im} \mu_A \subset \operatorname{Ker} u$. Let $\mathcal{B}$ be the right-angled Artin group with generators $\{h_e\}_{e \in E}$ and relations $h_e h_f h_e^{-1} h_f^{-1} = 1$ for all edges $e, f \in E$ having no common vertices. Choose in each even cycle s in Γ a half $s' \subset s$. The formula $u(g_s) = \prod_{e \in s \setminus s'} h_e^{-1} \prod_{e \in s'} h_e$ defines a homomorphism $u : \mathcal{A} \to \mathcal{B}$. We claim that $u\mu_A = 1$ where $A \in \mathcal{D}$ and $\mu_A : \pi_1(X_{\mathcal{D}}, A) \to \mathcal{A}$ is the typing homomorphism determined by the orientation on $\mathcal{D}$ associated with the family $\{s'\}_s$. To see this, we compute $\mu = \mu_A$ on a path α in $X_{\mathcal{D}}$ using (5.1.1). Observe that if $A_k \in \mathcal{D}$ is the terminal endpoint of e_k and the initial endpoint of e_{k+1}, then for all k,

$$u(g_{|e_k|}^{\nu_k}) = \prod_{e \in A_{k-1}} h_e^{-1} \prod_{e \in A_k} h_e.$$

Multiplying these formulas over all k, we obtain $u\mu_A(\alpha) = \prod_{e \in A_0} h_e^{-1} \prod_{e \in A_n} h_e$. For $A_0 = A_n = A$, this gives $u\mu_A(\alpha) = 1$.

9.3. **Generators and relations.** For a dimer covering $A \in \mathcal{D}$ and a vertex v of Γ, denote the only edge of A incident to v by A_v. A triple $A, B, C \in \mathcal{D}$ is *flat* if for any vertex v of Γ, at least two of the edges A_v, B_v, C_v are equal.

Theorem 9.3. *For each $A_0 \in \mathcal{D}$, the dimer group $\pi_1(X_{\mathcal{D}}, A_0)$ is isomorphic to the group with generators $\{y_{A,B}\}_{A,B\in\mathcal{D}}$ subject to the following relations: $y_{A,C} = y_{A,B}\, y_{B,C}$ for each flat triple $A, B, C \in D$ and $y_{A_0,A} = 1$ for all $A \in \mathcal{D}$.*

Proof. This theorem follows from Lemma 9.1, Theorem 6.1, and the following claim: three dimer coverings $A, B, C \in \mathcal{D}$ are vertices of a cube in $\mathcal{D}$ if and only if the triple A, B, C is flat. We now prove this claim.

Suppose that the triple A, B, C is flat. Then every vertex of Γ is incident to a unique edge belonging to at least two of the sets $A, B, C \subset E$. Such edges form a dimer covering, K, of Γ. For any vertex v of Γ, either $A_v = B_v = C_v = K_v$ or $A_v = B_v = K_v \neq C_v$ up to permutation of A, B, C. In the first case, the sets $AK, BK, CK \subset E$ contain no edges incident to v. In the second case, AK, BK contain no edges incident to v while CK contains two such edges C_v and K_v. This shows that AK, BK, CK are pairwise independent cyclic sets of edges. They split (uniquely) as unions of independent cycles. All these cycles are even: their intersections with K are their halves. Denote the resulting sets of even cycles by X, Y, Z, respectively. Thus, $AK = [X], BK = [Y], CK = [Z]$. Equivalently, $A = [X]K$, $B = [Y]K$, and $C = [Z]K$. Then the cube in $\mathcal{D}$ determined by the based cube $(K, X \cup Y \cup Z)$ contains A, B, and C.

Conversely, suppose that $A, B, C \in \mathcal{D}$ are vertices of a cube in $\mathcal{D}$. By the last remark of Section 6.1, there is $K \in \mathcal{D}$ and a set of independent even cycles S with a partition $S = X \amalg Y \amalg Z$ such that $A = [X]K$, $B = [Y]K$, and $C = [Z]K$. Consider a vertex v of Γ. If v is not incident to the edges forming the cycles of S, then $A_v = B_v = C_v = K_v$. If v is incident to an edge belonging to a cycle $s \in X$, then the edges forming the cycles of Y and Z are not incident to v and $B_v = C_v = K_v$. The cases $s \in Y$ and $s \in Z$ are similar. In all cases, at least two of the edges A_v, B_v, C_v are equal. Therefore, the triple A, B, C is flat. □

Corollary 6.4 yields the following computation of the *dimer groupoid* $\pi_1(X_{\mathcal{D}}, \mathcal{D})$.

Theorem 9.4. *The groupoid $\pi_1(X_{\mathcal{D}}, \mathcal{D})$ is presented by generators $\{z_{A,B} : A \to B\}_{A,B\in\mathcal{D}}$ and relations $z_{A,C} = z_{A,B}\, z_{B,C}$ for every flat triple $A, B, C \in \mathcal{D}$.*

9.4. **The space $L(\Gamma)$.** We relate the dimer complex $X_{\mathcal{D}} \subset X_G$ to the space $L(\Gamma)$ of dimer labelings of Γ. In the sequel, non-even cycles in Γ are said to be *odd*.

Theorem 9.5. *Let $\omega : X_G \to I^E$ be the evaluation map from Section 7.3. The restriction of ω to $X_{\mathcal{D}}$ is an embedding whose image is a path-connected component $L_0(\Gamma)$ of the space of dimer labelings $L(\Gamma) \subset I^E$. The component $L_0(\Gamma)$ contains all dimer labelings associated with dimer coverings of Γ.*

Proof. The map ω carries a dimer covering of Γ into the associated dimer labeling of Γ. Therefore $\omega(X_{\mathcal{D}})$ contains the set $\mathcal{D}$ viewed as a subset of $L(\Gamma)$. Since $X_{\mathcal{D}}$ is path connected, $\omega(X_{\mathcal{D}})$ is contained in a path connected component, L_0, of $L(\Gamma)$. We shall show that the restriction of ω to $X_{\mathcal{D}}$ is a homeomorphism onto L_0.

Observe that $\mathcal{E} = \mathcal{D} \subset G$ satisfies the conditions of Lemma 7.2. Condition (i) holds because a subset of E may split as a union of independent cycles in at most one way. The partition of even cycles into halves satisfies (ii). Lemma 7.2 implies that the restriction of ω to $X_{\mathcal{D}}$ is injective.

Consider a dimer labeling $\ell : E \to I = [0,1]$. Then $\ell^{-1}((0,1)) \subset E$ is a cyclic set of edges. It splits uniquely as a (disjoint) union of $n \geq 0$ independent cycles $s_1, ..., s_n$. If s_i is odd for some i, then $\ell(s_i) = \{1/2\}$ (otherwise, s_i could be partitioned into edges with $\ell < 1/2$ and edges with $\ell > 1/2$ and would be even). Then, any deformation of ℓ in $L(\Gamma)$ preserves $\ell(s_i)$. So, $\ell \notin L_0$. Suppose that all the cycles $s_1, ..., s_n$ are even. We verify that $\ell \in \omega(X_{\mathcal{D}}) \subset L_0$. This will imply that $L_0 = \omega(X_{\mathcal{D}})$. For each $i = 1, ..., n$, pick a half, s'_i, of s_i. The definition of a dimer labeling implies that ℓ takes the same value, $x_i \in (0,1)$, on all edges belonging to s'_i and the value $1 - x_i$ on all edges belonging to the complementary half $s_i \setminus s'_i$. Set $A = \ell^{-1}(\{1\}) \cup \cup_{i=1}^n s'_i \subset E$. Clearly, the edges belonging to A have no common vertices. Since each vertex of Γ is incident to an edge with positive label, it is incident to an edge belonging to A. Therefore $A \in \mathcal{D}$. Since $s_1, ..., s_n$ are independent even cycles and $A \cap s_i = s'_i$ for each i, all vertices of the based cube $(A, S = \{s_1, ..., s_n\})$ belong to $\mathcal{D}$. The triple $(A, S, x : S \to I)$, where $x(s_i) = 1 - x_i$ for all i, represents a point $a \in X_{\mathcal{D}}$. It is easy to check that $\omega(a) = \ell$. So, $\ell \in \omega(X_{\mathcal{D}})$. □

Theorem 9.6. *The component $L_0(\Gamma)$ of $L(\Gamma)$ is homeomorphic to the dimer complex of Γ. All other components of $L(\Gamma)$ are homeomorphic to the dimer complexes of certain subgraphs of Γ.*

Proof. The first claim follows from Theorem 9.5. Arbitrary components of $L(\Gamma)$ can be described as follows. Consider a set S of independent odd cycles in Γ. Deleting from Γ the edges belonging to these cycles, their vertices, and all the edges of Γ incident to these vertices, we obtain a subgraph Γ_S of Γ. Each dimer labeling of Γ_S extends to a dimer labeling of Γ assigning $1/2$ to the edges belonging to the cycles in S and 0 to all other edges of Γ not lying in Γ_S. This defines an embedding $i : L(\Gamma_S) \hookrightarrow L(\Gamma)$. The proof of Theorem 9.5 shows that the image of i is a union of connected components of $L(\Gamma)$. In particular, $i(L_0(\Gamma_S))$ is a component of $L(\Gamma)$. Moreover, every component of $L(\Gamma)$ is realized as $i(L_0(\Gamma_S))$ for a unique S. (In particular, $L_0(\Gamma)$ corresponds to $S = \emptyset$.) It remains to note that $L_0(\Gamma_S)$ is homeomorphic to the dimer complex of Γ_S. □

Corollary 9.7. *All components of $L(\Gamma)$ are homeomorphic to non-positively curved cubed complexes and are aspherical. Their fundamental groups are isomorphic to the dimer groups of certain subgraphs of Γ.*

9.5. **Remarks.** 1. Consider finite graphs Γ_1, Γ_2 admitting dimer coverings. Let Γ' be obtained from $\Gamma = \Gamma_1 \amalg \Gamma_2$ by adding an edge connecting a vertex of Γ_1 with a vertex of Γ_2. Then $D(\Gamma') = D(\Gamma)$ and $X_{\mathcal{D}}(\Gamma') = X_{\mathcal{D}}(\Gamma)$. This implies that for any finite graph, there is a connected finite graph with the same dimer group.

2. If a finite graph has no dimer coverings, then one can subdivide some of its edges into two subedges so that the resulting graph has dimer coverings. Subdivision of edges into 3 or more subedges is redundant. This is clear from the fact that if a graph Γ' is obtained from a finite graph Γ by adding two new vertices inside the same edge, then there is a canonical homeomorphism $L(\Gamma) \approx L(\Gamma')$ carrying $L_0(\Gamma)$ onto $L_0(\Gamma')$ and $\mathcal{D}(\Gamma)$ onto $\mathcal{D}(\Gamma')$.

3. The typing homomorphism $\mu : D(\Gamma) \to \mathcal{A}$ of Section 9.2 induces an algebra homomorphism $\mu^* : H^*(\mathcal{A}) \to H^*(D(\Gamma))$ which may provide non-trivial cohomology classes of $D(\Gamma)$ (with coefficients in any commutative ring). The algebra $H^*(\mathcal{A})$ can be computed from the fact that the cells of the Salvetti complex appear in the

form of tori and so, the boundary maps in the cellular chain complex are zero (see, for example, [Ch]). The equality $u\mu = 1$ in Section 9.2 shows that μ^* annihilates $u^*(H^*(\mathcal{B})) \subset H^*(\mathcal{A})$. It would be interesting to know whether $\operatorname{Ker} \mu^*$ can be bigger than the ideal generated by $u^*(H^*(\mathcal{B}))$.

4. The proof of Theorem 9.6 shows that connected components of $L(\Gamma)$ bijectively correspond to sets of independent odd cycles in Γ.

10. Dimers vs. braids

The braid groups of a finite graph Γ share many properties of the dimer group $D(\Gamma)$. They are realizable as fundamental groups of non-negativelty curved complexes, and they embed in a finitely generated right-angled Artin group, see [Ab], [CW]. We construct a family of homomorphisms from $D(\Gamma)$ to the braid groups of Γ. We begin by recalling the braid groups of CW-spaces.

10.1. **Braid groups.** For a topological space X and an integer $n \geq 1$, the *ordered n-configuration space* $\tilde{C}_n = \tilde{C}_n(X) \subset X^n$ is formed by n-tuples of pairwise distinct points of X. The symmetric group S_n acts on $\tilde{C}_n$ by permutations of the coordinates, and the quotient $C_n = \tilde{C}_n/S_n$ is the *unordered n-configuration space* of X. If X is a connected CW-space, then C_n is path connected. The group $B_n(X) = \pi_1(C_n)$ is the *n-th braid group* of X. The covering $\tilde{C}_n \to C_n$ determines (at least up to conjugation) a homomorphism $\sigma_n : B_n(X) \to S_n$.

10.2. **V-orientations.** Given a cycle s in a graph Γ, denote by ∂s the set of vertices of s, i.e., the set of vertices of Γ incident to the edges belonging to s. Clearly, $\operatorname{card}(s) = \operatorname{card}(\partial s)$. If s is even, then ∂s has a partition into two subsets obtained by following along the circle $\underline{s}$ and collecting all odd-numbered vertices into one subset and all even-numbered vertices into another subset. These subsets are called *vertex-halves* or, shorter, *v-halves* of s. A *v-orientation* of Γ is a choice of a distinguished v-half in each even cycle in Γ. If k is the number of even cycles in Γ, then Γ has 2^k v-orientations.

10.3. **The map $\Theta : L_0(\Gamma) \to C_N(\Gamma)$.** Let Γ be a v-oriented finite graph admitting at least one dimer covering. All dimer coverings of Γ have the same number, N, of edges equal to half of the number of vertices of Γ (the latter number has to be even). Let E be the set of edges of Γ. We define a map Θ from the dimer space $L_0(\Gamma) \subset I^E$ to the unordered N-configuration space $C_N(\Gamma)$. To this end, we parametrize all edges of Γ by the interval $I = [0, 1]$. This allows us to consider convex combinations of points of an edge. The homotopy class of Θ will not depend on the parametrizations.

For a dimer labeling $\ell \in L_0(\Gamma)$, we define an N-point set $\Theta(\ell) \subset \Gamma$ as follows. Each edge $e \in \ell^{-1}(\{1\}) \subset E$ contributes to $\Theta(\ell)$ the mid-point $(a_e + b_e)/2 \in e$ where a_e, b_e are the vertices of e. Other points of $\Theta(\ell)$ arise from the cyclic set of edges $\ell^{-1}((0, 1)) \subset E$. This set is formed by several independent cycles $s_1, ..., s_n \subset E$. The inclusion $\ell \in L_0(\Gamma)$ guarantees that these cycles are even, cf. the proof of Theorem 9.5. Each s_i contributes $\operatorname{card}(s_i)/2$ points to $\Theta(\ell)$. If $\ell(s_i) = \{1/2\}$, then these points are the vertices of s_i belonging to the distinguished v-half. If $\ell(s_i) \neq \{1/2\}$, then ℓ takes the same value $x_i \in (1/2, 1)$ on every second edge of s_i. Each such edge, e, has a vertex, a_e, in the distinguished v-half of s_i and a vertex, b_e, in the complementary v-half. We include in $\Theta(\ell)$ the point of e represented by the convex combination $(3/2 - x_i)a_e + (x_i - 1/2)b_e$. The coefficients are chosen so that

when x_i converges to $1/2$ the combination converges to a_e, and when x_i converges to 1 the combination converges to $(a_e + b_e)/2$. Note that the points contributed by s_i lie on the circle $\underline{s}_i \subset \Gamma$. Therefore all selected points are pairwise distinct and $\mathrm{card}(\Theta(\ell)) = N$. This defines a continuous map $\Theta : L_0(\Gamma) \to C_N(\Gamma)$.

10.4. **The homomorphism θ.** Let Γ, N, E be as in Section 10.3 with connected Γ. Then $C_N(\Gamma)$ is connected and the map $\Theta : L_0(\Gamma) \to C_N(\Gamma)$ induces a homomorphism of the fundamental groups $\theta = \theta(\Gamma) : D(\Gamma) \to B_N(\Gamma)$. This homomorphism is not necessarily injective and may be trivial, see Example 10.7 below.

The composition of θ with $\sigma_N : B_N(\Gamma) \to S_N$ can be explicitly computed as follows. Pick a dimer covering A of Γ and mark the edges belonging to A by (distinct) numbers $1, 2, ..., N$. A loop in $X_{\mathcal{D}}(\Gamma)$ based at A is represented by a sequence of consecutive glidings of A along certain even cycles $s_1, ..., s_n$. Recursively in $i = 1, ..., n$, we accompany the i-th gliding with the transformation of the marked dimer covering which keeps the marked edges not belonging to s_i and pushes each marked edge in s_i through its vertex belonging to the distinguished v-half of s_i. After the n-th gliding, we obtain the same dimer covering A with a new marking. The resulting permutation of the set $\{1, ..., N\}$ is the value of $\sigma_N\theta : D(\Gamma) = \pi_1(X_{\mathcal{D}}(\Gamma), A) \to S_N$ on the loop. The following example shows that, generally speaking, $\sigma_N\theta \neq 1$ and $\sigma_N\theta$ depends on the v-orientation of the graph.

10.5. **Example.** Consider the graph Γ in Figure 1 with vertices a, b, c, d, e, f. This graph has three cycles s_1, s_2, s_{12} formed, respectively, by the edges of the left square, by the edges of the right square, by all edges except the middle vertical edge. These cycles are even. We distinguish the v-halves, respectively, $\{a, e\}$, $\{c, e\}$, $\{b, d, f\}$. The vertical edges of Γ form a dimer covering, A. The group $D(\Gamma) = \pi_1(X_{\mathcal{D}}(\Gamma), A)$ is an infinite cyclic group with generator t represented by the sequence of glidings $A \mapsto s_1A \mapsto s_{12}s_1A \mapsto s_2s_{12}s_1A$. To compute $\sigma_3\theta : D(\Gamma) \to S_3$, we mark the edges of A with 1, 2, 3 from left to right. The transformations of A under the glidings are shown in Figure 1. Therefore $\sigma_3\theta(t) = (231)$. The opposite choice of the distinguished v-half in s_2 gives $\sigma_3\theta(t) = (213)$.

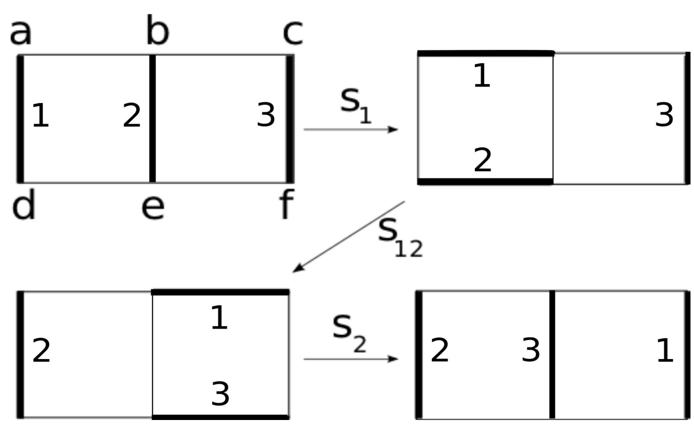

FIGURE 1. Transformations of a marked dimer covering

10.6. **The homomorphisms θ_n.** The homomorphism $\theta : D(\Gamma) \to B_N(\Gamma)$ of Section 10.4 can be included into a family of homomorphisms numerated by maps $n : E \to \{0, 1, 2, ...\}$. Set $|n| = \sum_{e\in E} n(e)$. Let Γ_n be the graph obtained from Γ by adding $2n(e)$ new vertices inside each edge e. The canonical bijection $\mathcal{D}(\Gamma) \approx \mathcal{D}(\Gamma_n)$ (cf. Remark 9.5.2) extends to a homeomorphism $X_{\mathcal{D}}(\Gamma) \approx X_{\mathcal{D}}(\Gamma_n)$ which, in its turn, induces a canonical isomorphism $D(\Gamma) \cong D(\Gamma_n)$. The v-orientation of Γ induces a v-orientation of Γ_n: the distinguished v-half of a cycle in Γ_n is the one including the distinguished v-half of the corresponding cycle in Γ. We define a homomorphism $\theta_n : D(\Gamma) \to B_{N+|n|}(\Gamma)$ as the composition

$$D(\Gamma) \cong D(\Gamma_n) \xrightarrow{\theta(\Gamma_n)} B_{N+|n|}(\Gamma_n) = B_{N+|n|}(\Gamma).$$

For $n = 0$, we have $\theta_n = \theta$. Example 10.7 below exhibits v-oriented graphs such that $\theta_n = 1$ for all n. It would be interesting to find out whether all θ_n may be trivial for all v-orientations of a graph with non-trivial dimer group. Is it true that $\cap \cap_n \operatorname{Ker}(\sigma_{N+|n|}\theta_n) = 1$ where the first intersection runs over all v-orientations?

10.7. **Example.** Consider a finite connected graph Γ which is bipartite, i.e., the set of vertices of Γ is partitioned into two subsets V_0, V_1 such that every edge has one vertex in each. All cycles in Γ are even. We v-orient Γ by selecting in every cycle in Γ the v-half formed by the vertices belonging to V_0. Then the image of $\Theta : L_0(\Gamma) \to C_N(\Gamma)$ lies in $\prod_{v\in V_0} \Gamma_v \subset C_N(\Gamma)$ where $\Gamma_v \subset \Gamma$ is the union of all half-edges adjacent to v. Since Γ_v is contractible, the map Θ is homotopic to a constant map. Then $\theta = 1$. The graph Γ_n is also bipartite for all n, and $\theta_n = 1$. Other v-orientations in Γ may give non-trivial θ and θ_n, cf. Example 10.5.

11. Extension to hypergraphs

We extend the results of Section 9 to hypergraphs.

11.1. **Hypergraphs.** A *hypergraph* is a triple $\Gamma = (E, V, \partial)$ consisting of two sets E, V and a map $\partial : E \to 2^V$ such that $\partial e \neq \emptyset$ for all $e \in E$. The elements of E are *edges* of Γ, the elements of V are *vertices* of Γ, and ∂ is the *boundary map.* For $e \in E$, the elements of $\partial e \subset V$ are *vertices incident to* e or, shorter, *vertices of* e.

We briefly discuss examples of hypergraphs. A graph gives rise to a hypergraph in the obvious way. Every matrix M over an abelian group yields a hypergraph whose edges are non-zero rows of M, whose vertices are columns of M, and whose boundary map carries a row to the set of columns containing non-zero entries of this row. A CW-complex gives rise to a sequence of hypergraphs associated as above with the matrices of the boundary homomorphisms in the cellular chain complex. Coverings of sets by subsets also yield hypergraphs: a set E and a family of subsets $\{E_v \subset E\}_{v\in V}$ with $\cup_v E_v = E$ determine a hypergraph (E, V, ∂) where $\partial e = \{v \in V \,|\, e \in E_v\}$ for any $e \in E$.

11.2. **The gliding system.** Given a hypergraph $\Gamma = (E, V, \partial)$, we call two sets $s, t \subset E$ *independent* if $\partial s \cap \partial t = \emptyset$. Of course, independent sets are disjoint.

A *cyclic set of edges* in Γ is a finite set $s \subset E$ such that for every $v \in V$, the set $\{e \in s \,|\, v \in \partial e\}$ has two elements or is empty. A cyclic set of edges is a *cycle* if it does not contain smaller non-empty cyclic sets of edges.

Lemma 11.1. *If $s \subset E$ is a cyclic set of edges, then the cycles contained in s are pairwise independent and s is their (disjoint) union.*

Proof. Define a relation $\sim$ on s by $e \sim f$ if $e, f \in s$ satisfy $\partial e \cap \partial f \neq \emptyset$. This relation generates an equivalence relation on s; the corresponding equivalence classes are the cycles contained in s. This implies the lemma. $\square$

A cycle $s \subset E$ is *even* if s has a partition into two subsets called the *halves*, such that any two elements of the same half are carried by the boundary map ∂ to disjoint subsets of V. It is easy to see that if such a partition $s = s' \cup s''$ exists, then it is unique and $\cup_{e \in s'} \partial e = \cup_{e \in s''} \partial e$.

As in Section 8.2, even cycles in Γ in the role of glides together with the independence relation above form a regular set-like gliding system in the power group $G = 2^E$. By Corollary 3.5, the associated cubed complex $X_G = X_G(\Gamma)$ (with 0-skeleton G) is nonpositively curved. By Section 7.2, a choice of a distinguished element in each even cycle in Γ determines an orientation of G and a homomorphism $\mu_A : \pi_1(X_G, A) \to \mathcal{A}(G)$ for all $A \in G$. If E is finite, then the right-angled Artin group $\mathcal{A}(G)$ is finitely generated and μ_A is an injection.

11.3. **Dimer coverings and labelings.** A *dimer covering* of a hypergraph $\Gamma = (E, V, \partial)$ is a set $A \subset E$ such that each vertex of Γ is incident to exactly one element of A. Equivalently, a dimer covering of Γ is a set $A \subset E$ such that the family $\{\partial e\}_{e \in A}$ is a partition of V. Let $\mathcal{D} = \mathcal{D}(\Gamma)$ be the set of all dimer coverings of Γ. The same arguments as in the proof of Lemma 8.2 show that $\mathcal{D} \subset G = 2^E$ satisfies the square condition and the 3-cube condition. The associated cubed complex $X_{\mathcal{D}} \subset X_G$ with 0-skeleton $\mathcal{D}$ is the *dimer complex* of Γ. This complex is nonpositively curved.

A *dimer labeling* of a hypergraph $\Gamma = (E, V, \partial)$ is a labeling of the edges of Γ by non-negative real numbers such that for every vertex of Γ, the labels of the incident edges sum up to give 1 and only one or two of these labels may be non-zero. The set $L(\Gamma)$ of dimer labelings of Γ is a closed subset of the cube I^E. We endow $L(\Gamma)$ with the induced topology. Each dimer covering of Γ determines a dimer labeling that carries the edges of the covering to 1 and all other edges to 0.

11.4. **Finite hypergraphs.** A hypergraph $\Gamma = (E, V, \partial)$ is *finite* if the sets E and V are finite. Then the dimer complex $X_{\mathcal{D}}$ is connected and its fundamental group is the *dimer group* of Γ. With these definitions, the content of Section 9 including all statements and arguments applies *verbatim* to finite hypergraphs. One should simply replace the word "graph" with "hypergraph" everywhere.

References

[Ab] A. Abrams, *Configuration spaces of colored graphs.* Geom. Dedicata 92 (2002), 185-194.

[Bo] R. Bocklandt, *Consistency conditions for dimer models.* arXiv:1104.1592.

[BH] M.R. Bridson, A. Haefliger, Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.

[Ch] R. Charney, *An introduction to right-angled Artin groups.* Geom. Dedicata 125 (2007), 141-158.

[CW] J. Crisp, B. Wiest, *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups.* Algebr. Geom. Topol. 4 (2004), 439-472.

[DaJ] M. W. Davis, T. Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups.* J. Pure Appl. Algebra 153 (2000), no. 3, 229–235.

[DuT] G. Duchamp, J.-Y. Thibon, *Simple orderings for free partially commutative groups.* Internat. J. Algebra Comput. 2 (1992), no. 3, 351-355.

[HaW] F. Haglund, D. T. Wise, *Special cube complexes.* Geom. Funct. Anal. 17 (2008), no. 5, 1551-1620.

[HsW] T. Hsu, D. T. Wise, *On linear and residual properties of graph products.* Michigan Math. J. 46 (1999), no. 2, 251-259.

[Ke] R. Kenyon, *The dimer model.* Exact methods in low-dimensional statistical physics and quantum computing, 341-361, Oxford Univ. Press, Oxford, 2010.

[NR1] G. Niblo, L. Reeves. *Groups acting on CAT(0) cube complexes.* Geom. Topol., 1 (1997), 1-7.

[NR2] G. Niblo, L. Reeves. *The geometry of cube complexes and the complexity of their fundamental groups.* Topology, 37(1998), 621-633.

[SW] M. Sageev, D.T. Wise, *The Tits alternative for CAT(0) cubical complexes.* Bull. London Math. Soc. 37 (2005), no. 5, 706-710.

[STCR] N. C. Saldanha, C. Tomei, M.A. Casarin Jr., D. Romualdo, *Spaces of domino tilings.* Discrete Comput. Geom. 14 (1995), no. 2, 207-233.

Vladimir Turaev
Department of Mathematics
Indiana University
Bloomington IN47405, USA
vturaev@yahoo.com